

# On the classical confinement of test particles to a thin 3-brane in the absence of non-gravitational forces

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The classical confinement condition of test particles to a brane universe in the absence of non-gravitational forces is transformed using the Hamilton-Jacobi formalism. The transformed condition provides a direct criterion for selecting in a cosmological scenario 5D bulk manifolds wherein it is possible to obtain confinement of trajectories to 4D hypersurfaces purely due to classical gravitational effects.

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## 1. Introduction

In the braneworld theories with non-compact extra dimensions it is postulated that particles and fields of the standard model are confined to the brane universe, while the graviton is assumed to propagate both in the bulk and in the brane. Within the classical framework of this scenario, confining a test particle to the brane eliminates the effects of extra dimensions rendering them undetectable. In general non-gravitational forces acting in the bulk and orthogonal to spacetime are needed in order to keep the test particles moving on the brane, the source of these confining forces being interpreted in different manners. If the notion of confinement must appear in any reasonable theory with non-compact extra dimensions non-gravitational forces can not be excluded *a priori*. Confinement due to oscillatory behaviour has been proved in five-dimensional relativity with two times<sup>1,2</sup>. Indeed, null paths of massless particles in 5D geodesic motion can appear in 4D as timelike paths of massive particles which undergo oscillations in the 5D dimension around the 4D hypersurface. More-

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over in the four-dimensional spacetime there appears a non-gravitational force which gives rise to the possibility of a cosmological variation of the rest masses of the particles with consequent departure from geodesic motion . However it may worth studying under which conditions confinement is possible without the introduction of force fields, other than gravity, living in the bulk. Employing a phase space analysis of the splitting from the 5D to the 4D geodesic motion Dahia, Romero, da Silva and Tavakol <sup>3,4</sup> investigated the possible confinement of particles and photons in the neighbourhood of a four-dimensional hypersurface in five-dimensional warped product spaces and found a form of toroidal confinement. This quasi-confinement, which is oscillatory and neutrally stable, is due to classical gravitational effects without requiring the presence of other non-gravitational mechanisms. Our approach differs in various aspects from the previous one, but we find the similar result, that is, if the bulk geometry satisfies some general conditions confinement of test particles to a thin 3-brane is possible without the presence of non-gravitational mechanisms. In this paper we shall consider a foliation of the bulk, each leaf of the foliation representing a brane, and we shall succeed in selecting 5D bulk metrics which allow to choose at least one brane of the foliation where confinement is possible without the introduction of non-gravitational forces. We shall deal with a five-dimensional bulk and the geodesic motion of test particles will happen in a 5D background with three-dimensional isotropy and homogeneity. The dynamics of test particles as observed in 4D is generally discussed on the basis of the splitting of the geodesic equation in 5D. As pointed out by Ponce de Leon <sup>5,6</sup> this process has some drawbacks with regard to the definition of the so called “fifth force”. To overcome these drawbacks Ponce de Leon has analyzed the 5D geodesic equation in terms of local basis vectors and, suitably redefining various quantities, obtained a more correct description of test particles trajectories. An alternative formalism was presented by Seahra and Wesson <sup>7-9</sup> who based their analysis on a covariant foliation of the 5D manifold using 3+1 spacetime slices orthogonal to the extra dimension and derived the form of the classical non-gravitational force required to confine particles to a 4D hypersurface. We shall start from these last results but, as suggested by Ponce de Leon <sup>10</sup> in a somewhat different context, we shall go on utilizing the Hamilton-Jacobi (H-J) method. Indeed, the H-J formalism, where one has to deal with a “scalar” equation, will prove to be adequate to study the geodesic motion of test particles. As a final result it will be possible to select, in a cosmological scenario, 5D bulk manifolds wherein the motion of test particles can be confined at least to one 4D hypersurface of the

foliation without the introduction of non-gravitational forces. In all the others 4D hypersurfaces of the foliation either the confinement is guaranteed by non-gravitational forces or, in their absence, an observer on the brane will perceive that test particles move subject to an extra force and with variable 4D rest mass, as described by a number of authors in the literature<sup>11–23</sup>.

*Conventions.* Throughout the paper the 5D metric signature is taken to be  $(+, +, +, -, \varepsilon)$  where  $\varepsilon$  can be  $+1$  or  $-1$  depending on whether the extra dimension is spacelike or timelike, while the choice of 4D metric signature is  $(+, +, +, -)$ . The spacetimes coordinates are labelled  $x^i = (r, \vartheta, \varphi)$ ,  $x^4 = t$ . The extra coordinate is  $x^5 = y$ . Bulk indices will be denoted by capital Latin letters and brane indices by lower Greek letters.

## 2. Confinement of 5D trajectories to 4D hypersurfaces

The study of higher dimensional particle motion will be performed in the foliating approach. In this section we shall give a concise description of the geometric construction and of the confinement condition obtained by Seahra and Wesson<sup>7–9</sup>. These authors considered a 5D manifold  $M$  described by the metric  $g_{AB}(x^A)$  and introduced a scalar function  $\ell = \ell(x^A)$  which defines a foliation of this manifold by a series of 4D hypersurfaces  $l = \text{constant}$  denoted by  $\Sigma_\ell$ . The 5D manifold was referred to as “the bulk” so each leaf of the foliation is “a brane”. The hypersurfaces  $\Sigma_\ell$  were assumed to have a normal vector given by

$$n^A = \varepsilon \Phi \partial_A \ell, \quad n_A n^A = \varepsilon \quad (1)$$

The scalar  $\Phi$  which normalizes  $n^A$  is known as the lapse function. The projector tensor  $h_{AB}$  from the bulk to the hypersurfaces is

$$h_{AB} = g_{AB} - \varepsilon n_A n_B \quad (2)$$

This tensor is symmetric and orthogonal to  $n^A$ . Each hypersurface  $\Sigma_\ell$  was mapped by a 4D coordinate system  $\{\tilde{x}^\alpha\}$ . The four basis vectors

$$e_\alpha^A = \frac{\partial x^A}{\partial \tilde{x}^\alpha} \quad \text{with} \quad n_A e_\alpha^A = 0 \quad (3)$$

are tangent to the  $\Sigma_\ell$  hypersurfaces and orthogonal to  $n^A$ . These basis vectors can be used to project 5D objects onto  $\Sigma_\ell$  hypersurfaces. The induced metric on the  $\Sigma_\ell$  hypersurfaces

is given by

$$h_{\alpha\beta} = e_{\alpha}^A e_{\beta}^B g_{AB} = e_{\alpha}^A e_{\beta}^B h_{AB} \quad (4)$$

Clearly  $\{\tilde{x}^{\alpha}, \ell\}$  defines an alternative coordinate system to  $\{x^{\alpha}, y\}$  on  $M$ . 5D vectors were decomposed into the sum of a part tangent to  $\Sigma_{\ell}$  and a part normal to  $\Sigma_{\ell}$ . For  $dx^A$  it results

$$dx^A = e_{\alpha}^A d\tilde{x}^{\alpha} + (N^{\alpha} e_{\alpha}^A + \Phi n^A) d\ell \quad (5)$$

The 4D vector  $N^{\alpha}$  is called the shift vector and it describes how the  $\{\tilde{x}^{\alpha}\}$  coordinate system changes as one moves from a given  $\Sigma_{\ell}$  hypersurface to another. The 5D line element was then written as

$$ds_5^2 = h_{\alpha\beta} (d\tilde{x}^{\alpha} + N^{\alpha} d\ell) (d\tilde{x}^{\beta} + N^{\beta} d\ell) + \varepsilon \Phi^2 d\ell^2 \quad (6)$$

Finally we recall how the confinement condition was obtained in <sup>7-9</sup>. The equations of motion for a test particle moving in the bulk were given by

$$u^B \nabla_B u^A = \mathcal{F}^A, \quad u^A = \frac{d\tilde{x}^A}{d\lambda} \quad (7)$$

where  $\mathcal{F}$  is some non-gravitational force per unit mass and  $\lambda$  is a 5D affine parameter. These equations were decomposed into relations involving the particle velocity  $u^{\alpha} = e_{\alpha}^A u^A$  tangent to the  $\Sigma_{\ell}$  foliation and the particle velocity  $\xi = n_A u^A = \varepsilon \Phi \frac{d\ell}{d\lambda}$  parallel to the normal direction. In particular, the acceleration along the normal direction was found to be

$$u^A \nabla_A \xi = K_{\alpha\beta} u^{\alpha} u^{\beta} - \xi n^A u^B \nabla_A n_B + \mathcal{F}_n \quad (8)$$

where  $\mathcal{F}_n = n_A \mathcal{F}^A$  and  $K_{\alpha\beta}$  is the extrinsic curvature of the hypersurfaces  $\ell = \text{constant}$ :

$$K_{\alpha\beta} = e_{\alpha}^A e_{\beta}^B \nabla_A n_B \quad (9)$$

Now, if a test particle is confined to a given  $\Sigma_{\ell}$  hypersurface, its  $\ell$  coordinate must be constant. This implies that velocity and acceleration along the normal direction must vanish so eq. (8) with  $\xi = 0$  yields

$$K_{\alpha\beta} u^{\alpha} u^{\beta} + \mathcal{F}_n = 0 \quad (10)$$

which is the confinement condition in presence of non-gravitational forces obtained in <sup>7-9</sup>. In this work instead we shall require classical confinement due to gravitational forces alone, so eq. (10) will change to

$$K_{\alpha\beta} u^{\alpha} u^{\beta} = 0 \quad (11)$$

As discussed in <sup>7</sup>, this is a necessary condition for confinement which is a bilinear combinations between the components of the four-velocity and does not imply  $K_{\alpha\beta} = 0$  in general. If a member  $\Sigma_\ell$  of the foliation satisfies  $K_{\alpha\beta} = 0$ , which is termed the totally geodesic condition, then geodesic on that 4D hypersurface are also geodesic of the 5D manifold  $M$  <sup>24</sup>. In this case, as pointed out by Wesson <sup>25,26</sup>, the weak equivalence principle in 4D can be understood as a geometrical symmetry of 5D. Vice versa, if all the geodesics in  $\Sigma_\ell$  are also geodesic in  $M$ , then  $K_{\alpha\beta}$  is necessarily zero. Coming back to the purpose of the present paper, the constraint  $K_{\alpha\beta} u^\alpha u^\beta = 0$  requires the knowledge of the four-velocity  $u^\alpha$  which can be obtained, apart from mathematical difficulties, either from the spacetime components of the 5D geodesic equation or, as we shall do in this paper, from the H-J equation. If the constraint can be solved on a particular  $\Sigma_\ell$  hypersurface it seems reasonable, as conjectured also in <sup>10</sup>, to choose this hypersurface as the correct representation of our four-dimensional spacetime. If the constraint can not be solved an observer on that hypersurface will perceive the test particles move under the influence of an extra force with their 4D masses variable in time. In the above case with no confinement, if one adopts the most conservative point of view that confinement is a prerequisite for any reasonable theory with non-compact extra dimensions, then he has to introduce a non-gravitational confining force per unit mass given by  $\mathcal{F}^A = -(K_{\alpha\beta} u^\alpha u^\beta) n^A$ . In the next section we shall give a procedure for selecting 5D bulk manifolds wherein it is possible to achieve confinement in the absence of non-gravitational forces.

### 3. Confinement condition in the Hamilton-Jacobi formalism

Let us now return to the line element (6) and make some choices which will result in significant simplification of the following formulae. In braneworld theory the induced metric  $h_{\alpha\beta}$  is commonly identified with the spacetime metric  $g_{\alpha\beta}$ . In this paper we shall follow this approach and choose  $\tilde{x}^\alpha = x^\alpha$ . Moreover we shall select a “stationary” 4D coordinate frame setting  $N^\alpha = 0$  and let the other foliation parameter  $\Phi$  depend on the coordinates. In the cosmological scenario that we consider, our homogeneous and isotropic universe is embedded in a higher-dimensional manifold, so the 5D line element (6) will be rewritten in the usual form as

$$ds_5^2 = a^2(t, \ell) d\sigma^2 - n^2(t, \ell) dt^2 + \varepsilon \Phi^2(t, \ell) d\ell^2 \quad (12)$$

where

$$d\sigma^2 = dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (13)$$

$r, \vartheta, \varphi$  and  $t$  are the coordinates for a spacetime with spherically symmetric spatial sections while  $\ell = \ell(x^A)$  is the scalar function which defines the foliation of the 5D manifold. To solve the confinement condition (11) we shall not use the spacetime components of the 5D geodesic equation but, as suggested in a somewhat different context <sup>10</sup>, we shall consider the bulk geodesic motion of massive test particles by means of the Hamilton-Jacobi equation which in 5D is given by

$$g^{AB} \left( \frac{\partial S}{\partial x^A} \right) \left( \frac{\partial S}{\partial x^B} \right) = -m_5^2 \quad (14)$$

where  $m_5^2$  is the 5D rest mass of the particle and  $S(x^\alpha, \ell)$  is the five-dimensional action related to its 5D momentum by

$$P^A = g^{AB} \left( \frac{\partial S}{\partial x^B} \right) \quad (15)$$

Because of our choices on the spacetime metric, the 4D components of  $P^A$  are already projected onto a brane. We identify the affine parameter  $\lambda$  in (7) with the 5D proper time  $\tau_5$  and write  $u^\alpha = P^\alpha/m_5$  so in order to obtain the four-velocity we have to know the action  $S(x^\alpha, \ell)$ . In the case of massless particles the trajectory in 5D is along isotropic geodesics and is given by the Eikonal equation which can be obtained from the above formulae by setting  $m_5^2 = 0$  in the H-J equation (14) and substituting the momentum  $P^A$  in eq. (15) with the 5D wave vector  $k^A$ . In the massless case it will be therefore used  $k^\alpha$  instead of the velocity  $u^\alpha$ . In the spacetime with spherical symmetry which we consider, test particles move on a plane passing through the center. We take this plane as the  $\vartheta = \pi/2$  plane. Thus the H-J equation for the metric (12) is

$$\frac{1}{a^2} \left[ \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \varphi} \right)^2 \right] - \frac{1}{n^2} \left( \frac{\partial S}{\partial t} \right)^2 + \frac{1}{\varepsilon \Phi^2} \left( \frac{\partial S}{\partial \ell} \right)^2 = -m_5^2 \quad (16)$$

It is worth noticing that in the 4D proper time  $\tau_4$  parametrization <sup>7</sup> the four-velocity is  $v^\alpha = dx^\alpha/d\tau_4$  and the 4D momentum becomes  $P^\alpha = m_4 v^\alpha$  where  $m_4$  is the 4D rest mass. Therefore the H-J equation (16) gives the following relation between the rest mass in 4D and in 5D

$$m_4^2 = m_5^2 + \frac{1}{\varepsilon \Phi^2} \left( \frac{\partial S}{\partial \ell} \right)^2 \quad (17)$$

so the rest mass of a particle as perceived by an observer in 4D varies as a result of the 5D motion along the extra dimension, a result known in the literature that we shall obtain again below. The action  $S(x^\alpha, \ell)$  in (16) separates as

$$S(x^\alpha, \ell) = S_r(r) + L\varphi + S_{t\ell}(t, \ell) \quad (18)$$

where  $L$  is the angular momentum. Putting

$$\left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{L^2}{r^2} = C^2 \geq 0 \quad (19)$$

where  $C^2$  is a separation constant related to the motion in space, we finally obtain

$$\frac{1}{n^2} \left(\frac{\partial S_{t\ell}}{\partial t}\right)^2 - \frac{1}{\varepsilon \Phi^2} \left(\frac{\partial S_{t\ell}}{\partial \ell}\right)^2 = m_5^2 + \frac{C^2}{a^2} \quad (20)$$

Remembering (15), the particle velocity normal to  $\Sigma_\ell$  can be written as

$$\xi = \varepsilon \Phi \frac{d\ell}{d\tau_5} = \frac{1}{m_5 \Phi} \frac{\partial S_{t\ell}}{\partial \ell} \quad (21)$$

and the acceleration (8) becomes

$$u^A \nabla_A \xi = \frac{1}{m_5^2 \Phi} \left[ \frac{C^2}{a^3} \frac{\partial a}{\partial \ell} - \frac{1}{n^3} \frac{\partial n}{\partial \ell} \left(\frac{\partial S_{t\ell}}{\partial t}\right)^2 \right] + \frac{\xi}{m_5 n^2} \left(\frac{\partial S_{t\ell}}{\partial t}\right) \frac{\partial \ln \Phi}{\partial t} \quad (22)$$

We shall require that confinement becomes possible on a hypersurface  $\Sigma_0$  corresponding to  $\ell = \ell_0$  with  $a, n$  and  $\Phi$  finite and non-zero on this hypersurface. This implies that the velocity and the acceleration given, respectively, in (21) and (22) must vanish on  $\Sigma_0$  so we get

$$\left(\frac{1}{\Phi} \frac{\partial S_{t\ell}}{\partial \ell}\right)_{\ell=\ell_0} = 0 \quad (23a)$$

$$\left[\frac{C^2}{a^3} \frac{\partial a}{\partial \ell} - \frac{1}{n^3} \frac{\partial n}{\partial \ell} \left(\frac{\partial S_{t\ell}}{\partial t}\right)^2\right]_{\ell=\ell_0} = 0 \quad (23b)$$

Equation (23b) is the classical confinement condition (11) rewritten using the H-J formalism. The above conditions can be used not only to verify if a given bulk metric can lead to the required confinement but also to construct a new bulk metric with that property. In the former case one starts considering a known 5D metric and solves eq. (20). Then, if eqs. (23a) and (23b) are fulfilled, it means that in the bulk with the considered 5D metric confinement is possible in the absence of non-gravitational forces. In the latter case, as we shall show in

the following example, one can obtain particular 5D metrics satisfying the H-J equation and the confinement constraints. In detail, assuming that (23a) is satisfied by a still unknown function  $S_{t\ell}(t, \ell)$  and using (20), a simple way of satisfying (23b) is to put

$$n(t, \ell) = \frac{S_0 \frac{dF(t)}{dt}}{\sqrt{m_5^2 + \frac{C^2}{a^2(t, \ell)}}} \quad (24)$$

where  $S_0$  is a dimensionfull constant and  $F(t)$  is an arbitrary dimensionless function of the time  $t$ . Then it will prove useful to split eq. (20) into

$$\frac{1}{n} \left( \frac{\partial S_{t\ell}}{\partial t} \right) = \sqrt{m_5^2 + \frac{C^2}{a^2}} \cosh[\sqrt{\varepsilon} \beta(t, \ell)] \quad (25a)$$

$$\frac{1}{\sqrt{\varepsilon} \Phi} \left( \frac{\partial S_{t\ell}}{\partial \ell} \right) = \sqrt{m_5^2 + \frac{C^2}{a^2}} \sinh[\sqrt{\varepsilon} \beta(t, \ell)] \quad (25b)$$

where  $\beta(t, \ell)$  is a function to be determined and the Schwarz condition  $\partial^2 S_{t\ell} / \partial \ell \partial t = \partial^2 S_{t\ell} / \partial t \partial \ell$  has to be satisfied in the bulk. Clearly, in the case of massless particles one must require that  $C^2 \neq 0$ . Equation (23a) implies that  $\sinh[\sqrt{\varepsilon} \beta(t, \ell)] = 0$  on a particular hypersurface  $\Sigma_0$  for a value of  $\ell$  equal to a fixed  $\ell_0$ . Substituting (24) into (25a) we obtain

$$\left( \frac{\partial S_{t\ell}}{\partial t} \right) = S_0 \frac{dF(t)}{dt} \cosh[\sqrt{\varepsilon} \beta(t, \ell)] \quad (26)$$

The Schwarz condition is satisfied by the particularly simple choice where the function  $\beta(t, \ell)$  is dependent only on  $\ell$  in the form  $\beta(\ell) = \kappa(\ell - \ell_0)$ ,  $\kappa$  being a dimensionfull constant, and the fuction  $\Phi$  is given by

$$\Phi(t, \ell) = \frac{S_0 F(t) \kappa}{\sqrt{m_5^2 + \frac{C^2}{a^2(t, \ell)}}} \quad (27)$$

From (25b) we have

$$\left( \frac{\partial S_{t\ell}}{\partial \ell} \right) = S_0 F(t) \kappa \sqrt{\varepsilon} \sinh[\sqrt{\varepsilon} \kappa(\ell - \ell_0)] \quad (28)$$

therefore we obtain the function  $S_{t\ell}(t, \ell)$  as follows

$$S_{t\ell}(t, \ell) = S_0 F(t) \cosh[\sqrt{\varepsilon} \kappa(\ell - \ell_0)] \quad (29)$$

while the 5D line element (12) takes the form

$$ds_5^2 = a^2(t, \ell) \left[ d\sigma^2 + \frac{S_0^2}{C^2 + m_5^2 a^2(t, \ell)} \left( - \left( \frac{dF(t)}{dt} \right)^2 dt^2 + \varepsilon \kappa^2 F^2(t) d\ell^2 \right) \right] \quad (30)$$

the function  $a(t, \ell)$  and  $F(t)$  remaining arbitrary in this particular solution.



#### 4. Conclusions

We combined the geodesic and the Hamilton-Jacobi methods to select, in a cosmological scenario, 5D bulk metrics so that at least on one brane of the foliation of a 5D manifold  $M$  it is possible that test particles can be confined without the requirement of non-gravitational forces. Such a brane may represent our universe. Of course this does not exclude the possibility that confinement, if it must appear in any reasonable theory with non-compact extra dimensions, can be obtained also by means of non-gravitational mechanisms. Here we would like to briefly discuss some practical conditions which, once fulfilled, can ensure particle confinement. As a particularly simple example, a warped 5D metric of the form

$$ds_5^2 = e^{G(\ell)} \left[ a^2(t) d\sigma^2 - n^2(t) dt^2 \right] + \varepsilon \Phi^2(t, \ell) d\ell^2 \quad (31)$$

is apt to satisfy eq. (23b) provided that the warp factor  $e^{G(\ell)}$  is such that  $(dG(\ell)/d\ell)_{\ell=\ell_0} = 0$ , while a function  $S_{t\ell}(t, \ell)$  of the form  $S_{t\ell}(t, \ell) = S_0 F(t) G(\ell)$ , with the same  $G(\ell)$  of the warp factor, is apt to satisfy eq. (23a). We note that eq. (20), whose solution will give  $S_{t\ell}(t, \ell)$ , becomes more tractable if one isolates the effect of the extra dimension from the effects due to the motion in spacetime choosing the constant  $C^2 = 0$ . This simplification would however not be possible in the massless case ( $m_5^2 = 0$ ) if in eq. (20) also the term  $\frac{1}{\varepsilon \Phi^2} \left( \frac{\partial S}{\partial \ell} \right)^2$  should become equal to zero. Finally let us see whether conditions for confinement are met for the well known 5D metric describing a class of cosmological models found by Ponce de Leon <sup>27</sup>

$$ds_5^2 = \left( \frac{t}{L} \right)^{2/\alpha} \left( \frac{\ell}{L} \right)^{2/(1-\alpha)} d\sigma^2 - \left( \frac{\ell}{L} \right)^2 dt^2 + \left( \frac{\alpha}{1-\alpha} \right)^2 \left( \frac{t}{L} \right)^2 d\ell^2 \quad (32)$$

where  $L$  is a constant length and  $\alpha$  is a constant dimensionless parameter. The effects of the extra dimension as measured by an observer in 4D have been already examined in the background metric (32) but in a somewhat different context in <sup>19</sup>. We shall consider the following three possibilities:

**A)**  $m_5^2 > 0, C^2 = 0$ .

The solution to eq. (20) is

$$S_{t\ell}(t, \ell) = \frac{\alpha m_5}{\sqrt{2\alpha - 1} L} t \ell \quad (33)$$

Condition (23a) is not satisfied, so confinement is not possible. The trajectory  $\ell = \ell(\tau_4)$  as a function of the 4D proper time  $\tau_4$  can be written as

$$\ell = L \left( \frac{\tau_4}{L} \right)^{-((\alpha-1)^2/\alpha^2)} \quad (34)$$

From (17) evaluated along the trajectory (34) we obtain in agreement with <sup>19</sup> the rest mass  $m_4$  as

$$m_4 = \frac{\alpha m_5}{\sqrt{2\alpha - 1}} \quad (35)$$

therefore  $m_4$  takes on different values in different cosmological “eras” marked by a particular choice for  $\alpha$ .

**B)**  $m_5^2 = 0, C^2 = 0$ .

The solution to eq. (20) is

$$S_{t\ell}(t, \ell) = S_0 \left( \frac{t}{L} \right)^A \left( \frac{\ell}{L} \right)^{A\alpha/(\alpha-1)} \quad (36)$$

where  $A$  is a constant dimensionless parameter. As stated above, in the massless case with  $C^2 = 0$  one must not require that  $\Phi^{-1}(\partial S_{t\ell}(t, \ell)/\partial \ell)$  vanishes so condition (23a), which otherwise would be satisfied at  $\ell = 0$ , can not be taken into account here and confinement is not possible. The trajectory  $\ell = \ell(\tau_4)$  as a function of the 4D proper time  $\tau_4$  can be written as

$$\ell = L \left( \frac{\tau_4}{\alpha L} \right)^{(1-\alpha)} \quad (37)$$

From (17) evaluated along the trajectory (37) we obtain the rest mass  $m_4$  as

$$m_4 = \frac{S_0 A \alpha}{\tau_4} \quad (38)$$

therefore, in agreement with <sup>19</sup>, the variation of  $m_4$  takes place in cosmological timescales.

**C)**  $m_5^2 = 0, C^2 > 0$ .

The solution to eq. (20) is

$$S_{t\ell}(t, \ell) = \frac{L C \alpha}{\sqrt{1 - 2\alpha}} \left( \frac{t}{L} \right)^{(\alpha-1)/\alpha} \left( \frac{\ell}{L} \right)^{\alpha/(\alpha-1)} \quad (39)$$

The left-hand side of condition (23a) does not vanish at  $\ell = 0$  because here  $\alpha < 1/2$ , so confinement is not possible. The trajectory  $\ell = \ell(\tau_4)$  as a function of the 4D proper time  $\tau_4$  can be written as

$$\ell = L \exp \left( - \frac{\tau_4}{L} \right) \quad (40)$$

From (17) evaluated along the trajectory (40) we obtain the rest mass  $m_4$  as

$$m_4 = \frac{C \alpha}{\sqrt{1 - 2\alpha}} \exp \left[ - \left( \frac{\alpha^2 + (1 - \alpha)^2}{1 - \alpha} \right) \left( \frac{\tau_4}{L} \right) \right] \quad (41)$$

therefore again  $m_4$  decreases on cosmological timescales. In all the three cases previously examined, confinement due only to classical gravitational effects is not possible.

We conclude by noting that there are some questions which need to be investigated with regard to the stability of the confined trajectories, but this will be the subject of a future work.

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- [1] A. P. Billyard and W. N. Sajko, *Gen. Rel. Grav.* **33** (2001) 1929 (gr-qc/0105074).
  - [2] P. S. Wesson, *Phys. Lett. B* **538** (2002) 159 (gr-qc/0205117).
  - [3] F. Dahia, C. Romero, L.P.F. Silva and R. Tavakol, *J. Math. Phys.* **48** (2007) 072501 (gr-qc/0702063).
  - [4] F. Dahia, C. Romero, L.P.F. Silva and R. Tavakol, *Gen. Rel. Grav.* **40** (2008) 1341 (gr-qc/0711.1279).
  - [5] J. Ponce de Leon, *Grav. Cosmol.* **8** (2002) 272 (gr-qc/0104008).
  - [6] J. Ponce de Leon, *Phys. Lett. B* **523** (2001) 311 (gr-qc/0110063).
  - [7] S. S. Seahra, *Phys. Rev. D* **65** (2002) 124004 (gr-qc/0204032).
  - [8] S. S. Seahra, *Phys. Rev. D* **68** (2003) 104027 (gr-qc/0309081).
  - [9] S. S. Seahra and P. S. Wesson, *Class. Quantum. Grav.* **20** (2003) 1321 (gr-qc/0302015).
  - [10] J. Ponce de Leon, *Int. J. Mod. Phys. D* **12** (2003) 757 (gr-qc/0209013).
  - [11] B. Mashhoon, P. S. Wesson and H. Liu, *Gen. Rel. Grav.* **30** (1998) 555.
  - [12] P. S. Wesson, B. Mashhoon, H. Liu and W. N. Sajko, *Phys. Lett. B* **456** (1999) 34.
  - [13] D. Youm, *Phys. Rev. D* **62** (2000) 084002 (hep-th/0004144).
  - [14] D. Youm, *Mod. Phys. Lett. A* **16** (2001) 2371 (hep-th/0110013).
  - [15] S. S. Seahra and P. S. Wesson, *Gen. Rel. Grav.* **33** (2001) 1731 (gr-qc/0105041).
  - [16] B. Mashhoon, H. Liu. and P. S. Wesson, *Phys. Lett. B* **331** (1994) 305.
  - [17] H. Liu and B. Mashhoon, *Phys. Lett. A* **272** (2000) 26 (gr-qc/0005079).
  - [18] F. Dahia, E. M. Monte and C. Romero, *Mod. Phys. Lett. A* **18** (2003) 1773 (gr-qc/0303044).
  - [19] J. Ponce de Leon, *Gen. Rel. Grav.* **35** (2003) 1365 (gr-qc/0207108).
  - [20] J. Ponce de Leon, *Gen. Rel. Grav.* **36** (2004) 1335 (gr-qc/0310078).
  - [21] J. E. M. Aguilar and M. Bellini, *Eur. Phys. J. C* **42** (2005) 349 (gr-qc/0408054).
  - [22] S. Jalalzadeh , B. Vakili and H. R. Sepangi, *Phys. Scr.* **76** (2007) 122 (gr-qc/040907).

- [23] O. Bertolami, C. G. Bömer, T. Harko and F. S. N. Lobo, *Phys. Rev. D* **75** (2007) 104016 (gr-qc/0704.1733).
- [24] H. Hishihara, *Phys. Rev. Lett.* **86** (2001) 381 (gr-qc/0007070).
- [25] P. S. Wesson, *Gen. Rel. Grav.* **35** (2003) 307 (gr-qc/0302082).
- [26] P. S. Wesson, *Int. J. Mod. Phys. D* **14** (2005) 2937 (gr-qc/0601065).
- [27] J. Ponce de Leon, *Gen. Rel. Grav.* **20** (1988) 539.